A GEOMETRIC APPROACH TO DEFINING MULTIPLICATION

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How can we physically interpret real number multiplication? Or to put it another way, is there a simple way to visualize multiplication of any two real numbers? For example, given any two line segments how could you create a third line segment that is the product of the first two segments? The area model for multiplication is the orthodox physical interpretation that is given to multiplication. Multiplication of two line segments is then viewed as the area of the rectangle they determine. However, for this model to make sense we must relate the area of a rectangle, which is a two-dimensional object, with a line segment, which is a one-dimensional object. In particular, under the area model, how to convert the area of a rectangle to a line segment is not easily visualizable (especially if the numbers being multiplied are not rational).

In this paper we will do the following: (1) show how to geometrically define multiplication, using only basic plane geometry, independently of area and any notion of similar triangles; (2) prove all the properties of multiplication using only the axioms of plane geometry and the geometric definition of multiplication; (3) explain how the geometric definition of multiplication relates to the area of a right triangle (or rectangle); and (4) explain how by using only the geometric definition of multiplication and the Pythagorean Theorem one can prove that two triangles have the same angles if and only if the lengths of their corresponding sides are proportional. The interesting and surprising thing, from a pedagogical and/or mathematical point of view, is that all of these results can be proven using only simple geometry (no limits needed). As we shall see, parallel lines in our geometric approach will play a role similar to limits in the standard algebraic approach.

This paper was written from the perspective of both a mathematician and math educator and is intended to be read (and discussed) by members in both communities. Unfortunately, we have been unable to find a journal whose audience includes both mathematicians and math educators. Not surprisingly, we have found that our colleagues, in both communities, do not quite know how to classify the paper. Consequently, the paper has been stuck in limbo somewhere between pure mathematics and pure mathematics education. It is our sincere hope that our colleagues in both communities will give the paper due consideration.

The area model for multiplication is the orthodox way of extending a student's understanding of multiplication from the whole numbers to the positive real numbers. Under this model each positive real number is viewed as a line segment. Multiplication of two line segments is then viewed as the area of the rectangle they determine. For this model to make sense we must relate the area of a rectangle, which is a two-dimensional object, with a line segment, which is a one-dimensional

object. When the sides of the rectangle are rational numbers this is accomplished by partitioning the rectangle into a finite number of squares with rational sides. However, when the sides of the rectangle are irrational numbers this method breaks down and the measure of the rectangle's area becomes a limit of an infinite series. Moreover, since the area model does not cover signed number multiplication, it is unclear how to multiply signed numbers. It follows, a rigorous definition of positive number multiplication as well as proofs of many of the basic properties of multiplication require limits and are, therefore, beyond the beginning student. While students may not be aware of any of these considerations, the underlying issues are confusing to them, as evidenced by difficulties with understanding signed number multiplication. Futhermore, the intuitive models students develop about multiplication early on, namely multiplication with whole numbers is repeated addition, continue to tacitly affect their thinking which can then lead to many misconceptions about multiplication. For example, two major misconceptions beginning students have with multiplication are: (1) multiplication always makes bigger; and, (2) *non-conservation of operations* (that is when students cannot solve word problems when *easy* numbers are replaced with *hard* numbers) (see [7]). The best ways of extending multiplication from the whole numbers to the integers, rational and decimal numbers remain a major pedagogical challenge and has been the focus of much research (see [7],[8] and [9]). The lack of an easy visualization of multiplication (how to multiply two line segments), in contrast with addition, may account for some of the struggles that students encounter when they first make the transition from the whole numbers to the integers and rational numbers. Indeed, when students are first introduced to multiplication with fractions and signed numbers, we and several of our colleagues believe, most students start to view mathematics more as an esoteric-magical religion rather than a deductive science.

The geometric definition we propose addresses the modeling needs of educators, and the rigor needs of mathematicians, within one context that does not require limits. Namely, it provides a visual and intuitive model for extending multiplication all the way from whole numbers to real numbers; and it does so in a mathematically rigorous way, with axiom-based proofs. It is our hope that this highly visual path to the mathematical rigor of multiplication, which lends itself well to dynamic geometry software, will be of interest from both a mathematical and pedagogical standpoint.

Seven particularly noteworthy features of our geometric definition of multiplication are: (1) it does not require Cauchy sequences or limits; (2) it integrates the views of multiplication as repeated addition and as scaling; (3) it can be used as a way to demystify signed number and fraction multiplication for beginning students; (4) it can be used to provide simple geometric proofs of the associative, commutative and distributive properties of multiplication; (5) it can be used to help resolve common misperceptions many students have about multiplication; (6) it gives students access to insights about key axiomatic structures for multiplication and (7) it is does not require area considerations or presuppose any knowledge of similar triangles.
It is interesting to note the potentially profound affect that switching foundations from algebra to geometry can have on a student's understanding of multiplication

and its properties.

It is essential for all Science, Technology, Engineering, and Mathematics (STEM) students to have a solid understanding of trigonometry and calculus - they are the language of science. We believe that currently in the mathematics curriculum, in the U.S. at least, the intimate relationships between multiplication, the area of a rectangle, and similar triangles are not properly addressed. At best, a student may have seen a specious proof by whole numbers (that is if it holds for whole numbers then it must be true for all real numbers) for some of these relationships. Moreover, how many university and college textbooks define similar triangles is misleading.

Here is a typical definition of similar triangles given in many university and college textbooks: "Two triangles are similar if the corresponding angles are equal and the lengths of the corresponding sides are proportional." Most students will automatically assume that the ratio of corresponding sides being in proportion is part of the definition of similar triangles. Under this definition it is unclear whether any two triangles that have their corresponding angles equal would be considered similar. Of course a proper definition for similar triangles should state: "Two triangles are similar if their corresponding angles are equal." The fact that in similar triangles the lengths of the corresponding sides are proportional is actually a nontrivial theorem. The foregoing orthodox definition of similar triangles is analogous to making the following definition about a right triangle: "a triangle is a right triangle if one of its angles is ninety degrees and the sum of the squares of the two smaller sides equals the square of the longer side." This definition, while logically correct, is pedagogically unsound since it embeds the Pythagorean Theorem within the definition.

One might argue that any misunderstanding about similar triangles would eventually be resolved if the student were to take more math classes. Unfortunately, this is not true. Indeed in the U.S. a student can get a PhD in pure mathematics without ever seeing a proof that similar triangles have proportional sides. These facts are greatly alarming, since the definition of integration is built on limits of areas of rectangles and the whole subject of trigonometry is built on the premise that in similar triangles the ratio of corresponding sides must be in proportion. Moreover, it is presently considered pedagogically unsound to present the beginning mathematics student (in the U.S at least, this includes all students in mathematics courses up to the junior-undergraduate level) with a rigorous definition for real number multiplication. Consequently, Science, Technology and Engineering students can graduate with undergraduate degrees without ever seeing a rigorous definition of the real numbers and multiplication.

Most undergraduate mathematics students struggle to understand the modern construction (either through equivalence classes of Cauchy sequences of rational numbers or Dedekind cuts (see [11], [12] and [13])) of the real numbers and its operations. As a result teachers of basic mathematics, who have at most an undergraduate degree in mathematics (this includes the vast majority of K-12 teachers), may have a hard time trying to incorporate their knowledge of multiplication and its properties into their lesson plans. We believe that by comparing and contrasting the paper's geometric approach with the area model prospective K-12 mathematics teachers

will develop a more profound understanding of multiplication and its properties. This in turn will have a beneficial effect on the students they will teach.

1. THE GEOMETRIC DEFINITION OF MULTIPLICATION

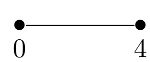
Throughout the paper we will freely assume Euclid's and/or Hilbert's axioms of plane geometry. We will now propose an alternative physical interpretation of real number multiplication which only uses parallel lines and is based on a simple observation about shadows (or projections). This basic observation is as follows: the hypotenuse of the right triangle determined by an object and its shadow must be parallel to the hypotenuse of any other object and its shadow. Hence, knowing the shadow of one object(we call this object the unit) gives us a way to deduce the shadow of any other object. Now if we replace object with line segment then we are led naturally to the following definition:

Definition 1. *Given two real numbers a and b lay a and b along the y- and x-axis respectively. We define ab (read "a multiplied by b") to be the x-intercept of the line parallel to the segment $\overline{(b,0),(0,1)}$ which passes through the point $(0,a)$.*

In this definition, $(0,1)$ is our unit, $(b,0)$ is our unit shadow determined by b, and segment $\overline{(b,0),(0,1)}$ is our unit hypotenuse determined by b. Please note this geometric definition of multiplication only uses parallel lines and **does not presuppose any knowledge of similar triangles. Moreover, to avoid circular reasoning all the proofs and examples based on the definition will only use congruency arguments.**

Example 1. *Use Definition 1 to multiply 2 and 4 then use congruent triangles to show that 2(4)=4+4.*

Solution:
Step 1:



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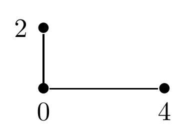

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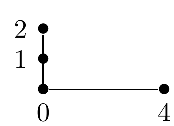
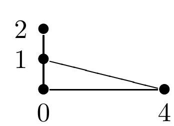
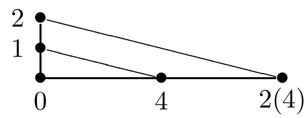


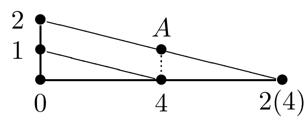
Step 4: Connect (0,1) to (4,0).



Step 5: Now draw the line passing through (0,2) which is parallel to the segment $\overline{(0,1),(4,0)}$. By Definition 1, the x-intercept of this line is 2(4).



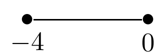
Step 6: Refer to the figure below. There exists a point, A, on $\overline{(0,2),(2(4),0)}$ such that $\overline{\{A,(4,0)\}}$ is parallel to $\overline{(0,1),(0,2)}$. By design $\{(0,1),(0,2),(4,0),A\}$ form the vertices of a parallelogram (why?). This implies $A=(4,1)$.



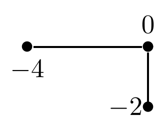
Using angle-side-angle we can deduce that the triangle determined by $\{(0,1),(0,0),(4,0)\}$ is congruent to the triangle determined by $\{(4,0),A,(2(4),0)\}$. Hence we must have 2(4)=4+4.

Example 2. *Use Definition 1 to multiply -2 and -4 then use congruent triangles to show that* $(-2)(-4)=2(4)$.

Solution:
Step 1:



Step 2:



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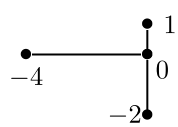
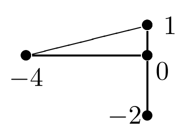
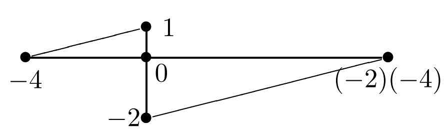


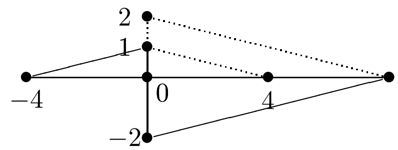
Step 4: Connect (0,1) to (-4,0).



Step 5: Now draw the line through (0,-2) which is parallel to the segment $\overline{(0,1),(-4,0)}$. By Definition 1, the x-intercept of this line is (-2)(-4).

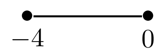


Step 6: In the figure below one can easily show, using congruent triangles and example 1, that (-2)(-4)=2(4).

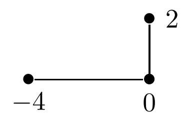


Example 3. *Use Definition 1 to multiply 2 and -4 then use congruent triangles to show that 2(-4)=-(2(4))=-8.*

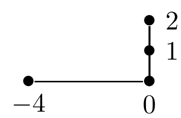
Solution:
Step 1:



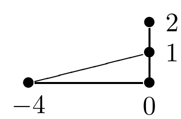
Step 2:



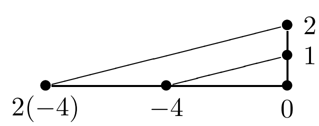
Step 3:



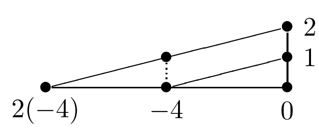
Step 4: Connect (0,1) to (-4,0).



Step 5: Now draw the line passing through (0,2) which is parallel to the segment $\overline{(0,1),(-4,0)}$. By Definition 1, the x-intercept of this line is 2(-4).



Step 6: Connect (-4,0) to (-4,1).



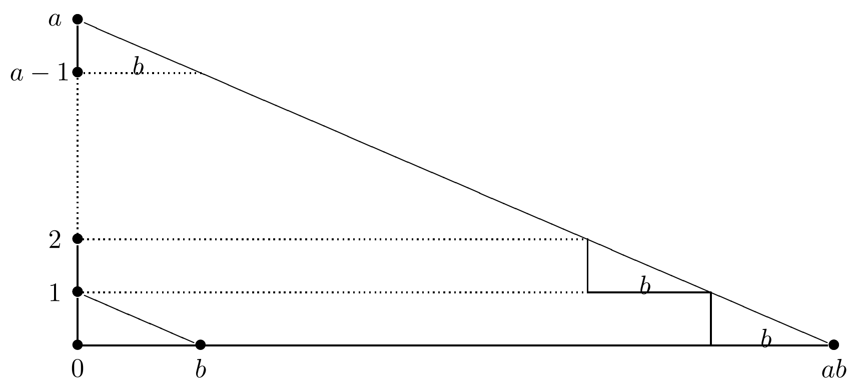
In the figure above the triangle determined by the three points (0,1), (0,0) and (-4,0) is congruent to the triangle determined by (-4,0), (-4,1) and (2(-4),0). Hence we must have $2(-4) = -(4+4) = -8$. It is left as an exercise for the reader to show, using definition 1, that $(-2)4 = 2(-4)$.
The foregoing examples provide the simple visual insight necessary in order to prove the general rules about sign number multiplication. Furthermore, example 1 provides the intuitive insight necessary to show that our multiplication definition reduces to repeated addition when restricted to whole numbers.

2. Extending students' understanding of multiplication

In this section we show that our geometric definition of multiplication naturally extends the students' understanding of multiplication from the whole numbers (where multiplication can be viewed as repeated addition) to the real numbers.

Theorem 1. *If $a \neq 0$ is a whole number and b any real number then $ab = \sum_{i=1}^{a} b$.*

Proof. By definition of multiplication $\overline{(b,0),(0,1)}$ is parallel to $\overline{(0,a),(ab,0)}$. Partition a into units. Each unit of this partition can be used to determine a right triangle along the segment $\overline{(0,a),(ab,0)}$ (refer to figure below). Moreover, by angle-side-angle, each of these triangles is congruent to the triangle determined by the points: (0,0), (0,1) and (b,0). It follows we must have $ab = \sum_{i=1}^{a} b$.



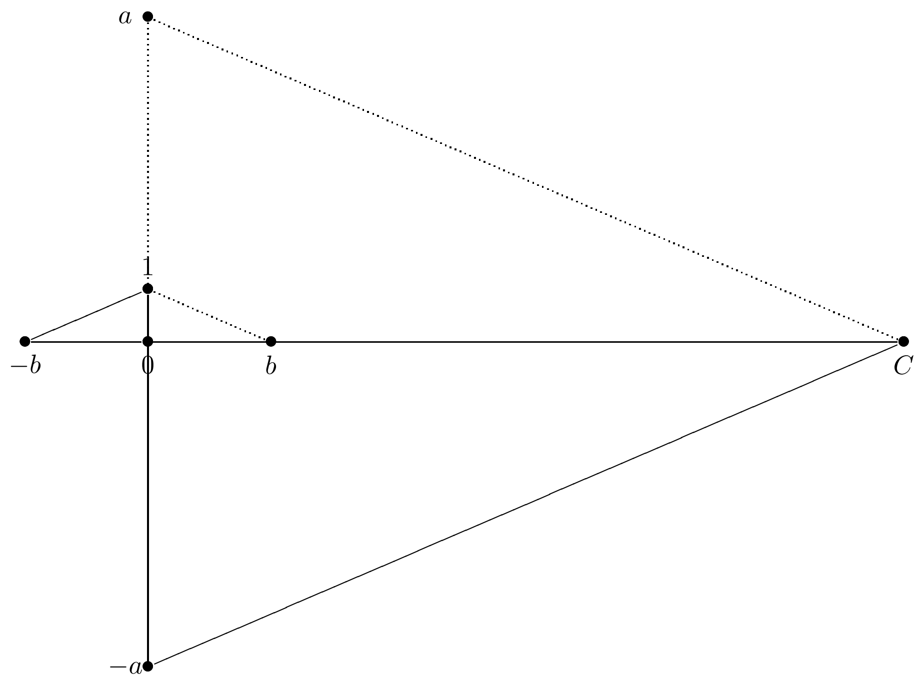
□

3. Multiplication of signed numbers

In this section we show that multiplication with signed numbers is completely natural. In particular, proving that a negative number times a negative number is a positive number (something most beginning students accept on faith) follows immediately from our definition of multiplication. In addition, our proof, unlike the traditional proof, does not require the use of the distributive property (for the conventional ways of teaching multiplication of signed numbers see [10]).

Theorem 2. *For any positive real numbers a and b the following are true:*
1.) $(-a)(-b) = ab$;
2.) $a(-b) = a(-b) = -(ab)$.

Proof. We prove one, the other case follows mutatis mutandis. In the figure below the two smaller triangles are congruent by side-angle-side. Moreover, the two larger triangles are also congruent by side-angle-side. This implies, $\overline{(b,0),(0,1)} \parallel \overline{C,(0,a)}$ if and only if $\overline{(-b,0),(0,1)} \parallel \overline{C,(0,-a)}$. Hence, by definition of multiplication, we must have $(-a)(-b) = ab$.



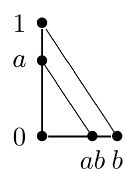
□

4. The relative size of Multiplied numbers

In this section we use our definition of multiplication to visually show that the product of two positive real numbers may not always be larger than the numbers being multiplied.

Theorem 3. *If $0 < a < 1$ and $b > 0$ then $ab < b$.*

Proof. Refer to the figure below. Proof follows directly from Definition 1.



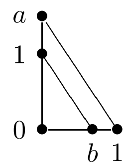
□

5. The Inverse of a real number

Theorem 4. *If $a \neq 0$ is a real number then there exists a unique real number $b \neq 0$ such that $ab = 1$. Moreover, we call b the inverse of a and denote it by $\frac{1}{a}$.*

Proof. Consider the unique line that is parallel to $\overline{(0,a),(1,0)}$ and passes through $(0,1)$. Let $(b,0)$ be the x-intercept of this line (refer to figure below). By definition

of multiplication we have $ab = 1$.



$\square$

6. Multiplication with rational numbers

In this section we define the rational numbers and then show how multiplication with rational numbers is related to whole number multiplication.

Definition 2. *Let $a \neq 0$ and b be real numbers. We define $\frac{b}{a} := b(\frac{1}{a})$. When $a \neq 0$ and b are whole numbers we call the set of all real numbers of the form $\frac{b}{a}$ or $-\frac{b}{a}$, the rational numbers.*

Lemma 1. *If a and b are natural numbers then $\frac{b}{ab} = \frac{1}{a}$.*

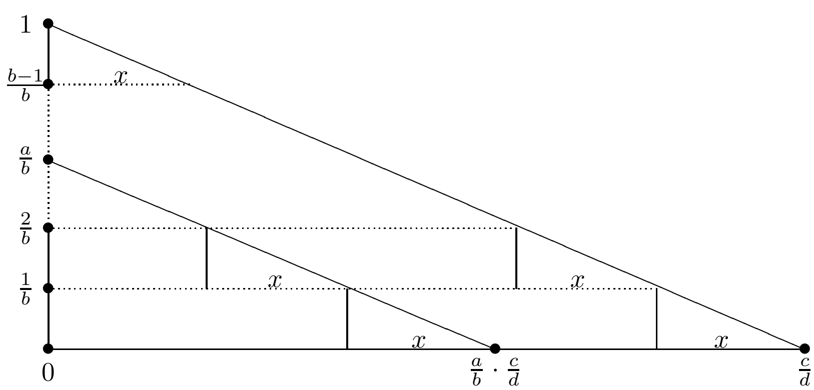
Proof. Note $1 = ab(\frac{1}{ab}) = a(b(\frac{1}{ab})) = a(\frac{b}{ab})$ and $1 = a(\frac{1}{a})$. Hence by Theorem 4 we must have $\frac{b}{ab} = \frac{1}{a}$ $\square$

Lemma 2. *If a,b and c are natural numbers then $\frac{ac}{ab} = \frac{c}{b}$.*

Proof. Note $\frac{ac}{ab} = c\left(\frac{a}{ab}\right)$ and $\frac{c}{b} = c\left(\frac{1}{b}\right)$. Hence, claim follows by Lemma 1. $\square$

Lemma 3. *If $a \neq 0$, b, c and $d \neq 0$ are integers then $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$. Moreover, $\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}$.*

Proof. We prove the case when a, b, c and d are natural numbers with $a < b$. All other cases follow mutatis mutandis. Partition $\overline{(0,1)}$ using $\frac{1}{b}$ (refer to figure below). This partition can be used to partition both $\overline{(0,1),(\frac{c}{d},0)}$ and $\overline{(0,\frac{a}{b}),(\frac{a}{b}\cdot\frac{c}{d},0)}$ into congruent right triangles (refer to figure below). Observe, from the figure below, that $bx = \frac{c}{d}$. By Lemmas 1 and 2, we can deduce that $x = \frac{c}{d \cdot b}$. Consequently, $ax = a\left(\frac{c}{d \cdot b}\right) = \frac{a \cdot c}{d \cdot b}$. Moreover, from the figure below, $ax = \frac{a}{b} \cdot \frac{c}{d}$. Hence, $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} = \frac{c \cdot a}{d \cdot b} = \frac{c}{d} \cdot \frac{a}{b}$.



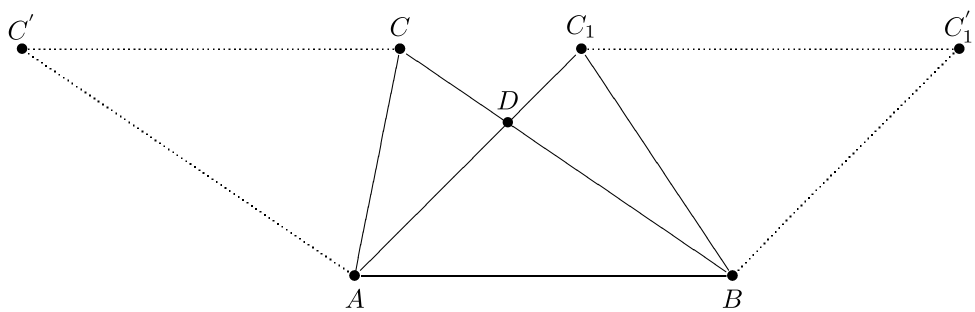
$\square$

7. An impetus for developing multiplication

The following Lemma is Propostion 37 in book one of Euclid's *elements*.

Lemma 4. $A(\Delta ABC) = A(\Delta ABC_1)$ *if and only if* $CC_1 \parallel AB$

Proof. Suppose $CC_1 \parallel AB$. Refer to the diagram below. Let $C^{'}$ be the unique point where $CC^{'} = AB$ and $CC^{'} \parallel AB$. Similarly, let $C_1^{'}$ be the unique point where $C_1C_1^{'} = AB$ and $C_1C_1^{'} \parallel AB$.

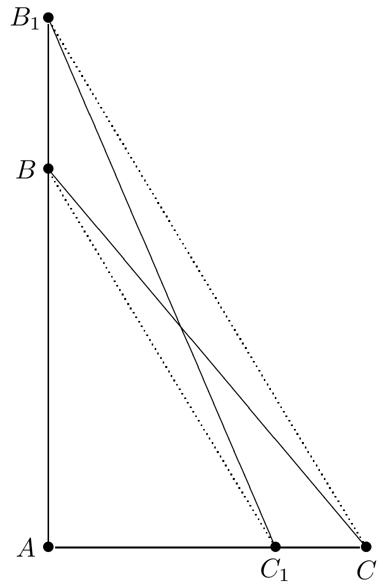


By side-angle-side we have $\Delta C^{'}CA = \Delta ABC$ and $\Delta ABC_1 = \Delta C_1^{'}C_1B$. This implies $C^{'}A = CB$ and $AC_1 = C_1^{'}B$. Now by side-side-side we have $\Delta C^{'}C_1A = \Delta CC_1^{'}B$. By referring to the diagram above we see that $A(\Delta C^{'}C_1A) - A(\Delta CDC_1) + A(\Delta ADB) = A(\Delta C^{'}CA) + A(\Delta ABC)$ and $A(\Delta CC_1^{'}B) - A(\Delta CDC_1) + A(\Delta ADB) = A(\Delta C_1^{'}C_1B) + A(\Delta ABC_1)$. Hence, from the foregoing statements we are able to deduce that $A(\Delta ABC) = A(\Delta ABC_1)$.
Conversely, suppose that $A(\Delta ABC) = A(\Delta ABC_1)$. Let $\tilde{C_1}$ be the unique point such that $C\tilde{C_1} \parallel AB$ and $C\tilde{C_1} \perp \tilde{C_1}C_1$ or $\tilde{C_1} = C_1$. By using the first part of the proof we have $A(\Delta ABC) = A(\Delta AB\tilde{C_1})$. It follows we must have $A(\Delta ABC_1) = A(\Delta AB\tilde{C_1})$ which is only possible if $\tilde{C_1} = C_1$. Therefore, we have shown that $CC_1 \parallel AB$. □

Theorem 5. *Let* ΔABC *and* ΔAB_1C_1 *be two right triangles with* $\angle BAC = \angle BAC_1 = 90$. *The areas of the two triangles are equal if and only if* BC_1 *is parallel to* B_1C.

Proof. Refer to figure below. Note $A(\Delta ABC) = A(\Delta AB_1C_1) \Leftrightarrow A(\Delta BC_1C) = A(\Delta B_1BC_1)$. Also by Lemma 4, $A(\Delta BC_1C) = A(\Delta B_1BC_1) \Leftrightarrow BC_1 \parallel B_1C$.

Hence claim follows.



□

Corollary 1. *Let ΔABC and ΔAB_1C_1 be two triangles with $\angle BAC = \angle BAC_1$. The areas of the two triangles are equal if and only if BC_1 is parallel to B_1C.*

Proof. Note there is nothing special about the angle being 90 degrees in the proof of Theorem 5. □

8. A relationship between the area of a right triangle and multiplication

A moment's reflection should convince the reader that the area of a triangle defines an equivalence relation on the set of all right triangles (this is a fancy way of saying it partitions the set of right triangles).

Question: Given a set containing all right triangles of the same area (an equivalence class) is it possible to assign it a real number in a well-defined way?

Answer: Yes. In each equivalence class there must be a right triangle that has the unit (some agreed upon length) as one of its legs (why?). We will take the length of the other leg of this triangle as the real number to assign to the equivalence class.

Question: Given any right triangle how can we find the real number that is assigned to its area?

Answer: We can use Theorem 5 to do this. In particular, given any right triangle, Theorem 5 provides a way to construct a second right triangle with the following properties: (1) it has the same area as the first triangle, and (2) the unit is one of its legs. The construction is a follows: let $(0, a)$ and $(b, 0)$ be the two legs of any right triangle then the legs of the second triangle can be taken as $(0, 1)$ and the x-intercept of the line parallel to the segment $\overline{(b, 0), (0, 1)}$ which passes through the point $(0, a)$ (note: this is precisely Definition 1).

Definition 3. *For positive real numbers a and b we define T_{ab} to be a right triangle with legs a and b. Moreover we will denote the area of T_{ab} by $A(T_{ab})$.*

Theorem 6. *$A(T_{ab}) = A(T_{a_1b_1})$ if and only if $ab = a_1b_1$.*

Proof. By definition of multiplication and Theorem 5 we have:
(1) $A(\Delta(0,0),(0,1),(ab,0)) = A(T_{ab})$;
(2) $A(\Delta(0,0),(0,1),(a_1b_1,0)) = A(T_{a_1b_1})$.
Now if $ab = a_1b_1$ then, by (1) and (2), we must have $A(T_{ab}) = A(T_{a_1b_1})$. Conversely, suppose $A(T_{ab}) = A(T_{a_1b_1})$. There exists a positive real number b_2 such that $ab = a_1b_2$. It follows $A(T_{ab}) = A(T_{a_1b_2})$ and $A(T_{ab}) = A(T_{a_1b_1})$ which imply $A(T_{a_1b_2}) = A(T_{a_1b_1})$. Hence, we must have $b_2 = b_1$ and $ab = a_1b_1$. □

Corollary 2. *Multiplication is commutative on positive real numbers.*

Proof. Note for real numbers a and b we have $A(T_{ab}) = A(T_{ba})$. By Theorem 6, we must have $ab = ba$. □

9. The Commutative Property of Multiplication

Theorem 7. *Multiplication of real numbers is commutative.*

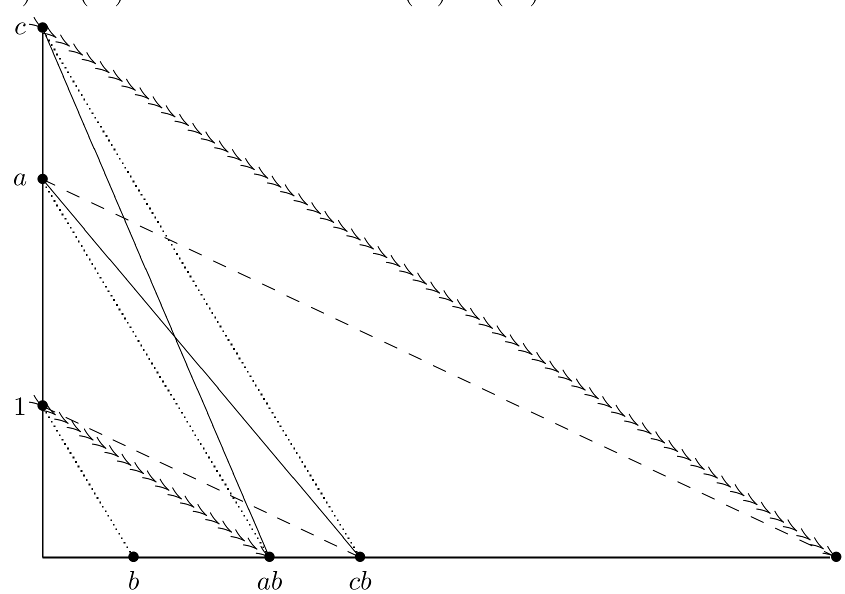
Proof. Claim follows from Corollary 2 and Theorem 2. □

10. The Associative Property of Multiplication

Definition 4. *For real numbers a,b and c we define $abc = a(bc)$.*

Lemma 5. *If a, b and c are real numbers then $a(cb) = c(ab)$.*

Proof. We will prove the case when $a > 0, b > 0$ and $c > 0$. The general case will then follow by Theorem 2. Refer to figure below. By definition of multiplication we have $\overline{(cb,0),(0,c)} \parallel \overline{(0,1),(b,0)}$ and $\overline{(ab,0),(0,a)} \parallel \overline{(0,1),(b,0)}$. This implies $\overline{(cb,0),(0,c)} \parallel \overline{(ab,0),(0,a)}$. By Theorem 5, $\overline{(cb,0),(0,c)} \parallel \overline{(ab,0),(0,a)}$ if and only if $A(T_{a(cb)}) = A(T_{c(ab)})$. Moreover, by Theorem 6, $A(T_{a(cb)}) = A(T_{c(ab)})$ if and only if $a(cb) = c(ab)$. Hence we must have $a(cb) = c(ab)$.



□

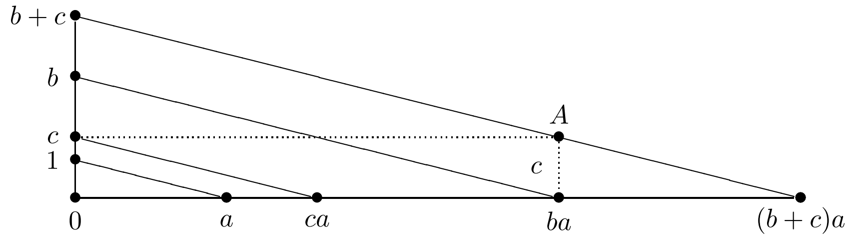
Theorem 8. *If a, b and c are real numbers then $a(bc) = (ab)c$.*

Proof. By Theorem 7, we have $a(bc) = a(cb)$ and $(ab)c = c(ab)$. This implies $a(bc) = (ab)c$ if and only if $a(cb) = c(ab)$. By Lemma 5 $a(cb) = c(ab)$. Hence we must have $a(bc) = (ab)c$. $\square$

11. The Distributive Property of Multiplication

Theorem 9. *$(b + c)a = ba + ca$ for all real numbers a,b and c.*

Proof. We prove the case when $a > 0$, $b > 0$ and $c > 0$. The other cases follow mutatis mutandis. By definiton of multiplication we have $\overline{(0, c), (ca, 0)} \parallel \overline{(0, b + c), ((b + c)a, 0)}$ (Why?). Refer to the figure below. By angle-side-angle we have triangle $\{(0, 0), (0, c), (ca, 0)\}$ congruent to triangle $\{(ba, 0), A, ((b + c)a, 0)\}$. This implies $(b + c)a - ba = ca$.



$\square$

12. Multiplication with fractions

We have choosen to write this section algebraically using the results from the previous sections, for two reasons. First, because the algebra is elegant enough, and second, to illustrate how the geometric definition of multiplication is in accord with the traditional algebraic definition and algorithms.

Lemma 6. *For nonzero real numbers b_1 and b_2 we have $\frac{1}{b_1} \cdot \frac{1}{b_2} = \frac{1}{b_1 \cdot b_2}$.*

Proof. Observe that $1 = (b_1 \cdot b_2)(\frac{1}{b_1 \cdot b_2})$. Moreover, by the associative and commutative properties of multiplication we have: $(b_1 \cdot b_2)(\frac{1}{b_1} \cdot \frac{1}{b_2}) = (b_1(\frac{1}{b_1})) \cdot (b_2(\frac{1}{b_2})) = 1$. Hence by uniqueness of inverses we must $\frac{1}{b_1} \cdot \frac{1}{b_2} = \frac{1}{b_1 \cdot b_2}$. $\square$

Theorem 10. *For real numbers a_1, a_2, $b_1 \neq 0$ and $b_1 \neq 0$ we have $\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} = \frac{a_1 \cdot a_2}{b_1 \cdot b_2}$.*

Proof. By the associative and commutative properties of multiplication we have: $\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} = (a_1(\frac{1}{b_1}))(a_2(\frac{1}{b_2})) = (a_1 \cdot a_2)(\frac{1}{b_1} \cdot \frac{1}{b_2})$. Moreover by Lemma 6 we have: $(a_1 \cdot a_2)(\frac{1}{b_1} \cdot \frac{1}{b_2}) = (a_1 \cdot a_2)(\frac{1}{b_1 \cdot b_2}) = \frac{a_1 \cdot a_2}{b_1 \cdot b_2}$. Hence we must have $\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} = \frac{a_1 \cdot a_2}{b_1 \cdot b_2}$. $\square$

Corollary 3. *For real numbers a, $b \neq 0$ and $k \neq 0$ we have $\frac{a}{b} = \frac{ak}{bk}$*

Proof. By Theorem 10 we have: $\frac{ak}{bk} = \frac{a}{b} \cdot \frac{k}{k} = \frac{a}{b} \cdot 1 = \frac{a}{b}$ $\square$

Corollary 4. *For real numbers a_1, a_2, $b_1 \neq 0$ and $b_1 \neq 0$ we have $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ if and only if $a_1 b_2 = a_2 b_1$.*

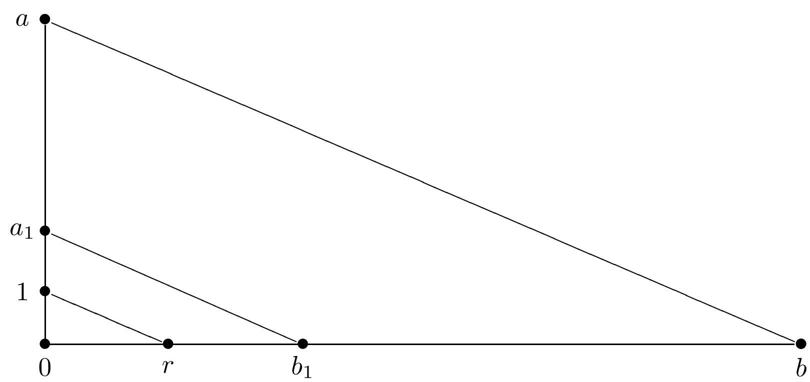
Proof. By Theorem 10 and Corollary 3 we have $a_1 b_2 = a_2 b_1 \Leftrightarrow (\frac{1}{b_1 b_2})(a_1 b_2) = (\frac{1}{b_1 b_2})(a_2 b_1) \Leftrightarrow \frac{a_1 b_2}{b_2 b_1} = \frac{b_1 a_2}{b_2 b_1} \Leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2}$ $\square$

13. Similar triangles

Definition 5. *We call two triangles similar if they have the same angles.*

Lemma 7. *If ΔABC is a right triangle with sides $a \leq b < c$ then any triangle $\Delta A_1B_1C_1$ with sides $a_1 \leq b_1 \leq c_1$ is similar to ΔABC if and only if there exists a $k > 0$ such that $a = ka_1, b = kb_1$ and $c = kc_1$.*

Proof. ($\Rightarrow$)
Suppose ΔABC is similar to $\Delta A_1B_1C_1$. Without-loss-of-generality we may assume $1 \leq a_1 < a$. There exists a $k > 1$ such that $a = ka_1$. Mark-off one unit along a_1. Next create a right triangle similar to ΔABC with legs 1 and r (refer to diagram below).



By definition of multiplication we have,
$b_1 = a_1r$ and $b = ar = k(a_1r) = kb_1$ (1).
Lastly by the Pythagorean Thm (we can use the Pythagorean Thm here since it can be proved using only area considerations),
$a_1^2 + b_1^2 = c_1^2$ and $a^2 + b^2 = c^2$ (2).
By (1) and (2) we must have $c^2 = (kc_1)^2$ or $c = kc_1$. Hence, claim follows.
($\Leftarrow$)
Suppose $a = ka_1, b = kb_1$ and $c = kc_1$. By the Pythagorean Thm, $a^2 + b^2 = c^2$ which implies $a_1^2 + b_1^2 = c_1^2$. Hence, $\Delta A_1B_1C_1$ must be a right triangle. By definition of multiplication, there exists a unique $r > 0$ such that $ar = b$ and $\overline{(0,a),(b,0)}$ is parallel to $\overline{(0,1),(r,0)}$. Now $ar = b, a = ka_1$ and $b = kb_1$ implies $ra_1 = b_1$. Therefore, by defintion of multiplication we have $\overline{(0,a_1),(b_1,0)}$ parallel to $\overline{(0,1),(r,0)}$. Hence, the triangles must be similar. $\square$

Theorem 11. *If ΔABC has sides $a \leq b \leq c$ then any triangle $\Delta A_1B_1C_1$ with sides $a_1 \leq b_1 \leq c_1$ is similar to ΔABC if and only if there exists a $k > 0$ such that $a = ka_1, b = kb_1$ and $c = kc_1$.*

Proof. Note any triangle can be cut into two right triangles. Hence claim follows by applying previous Lemma to each of the right triangles. $\square$

14. Conclusion

Mathematicians in the time of Pythagoras lived in a world where magnitudes and numbers were not the same thing (see [18]). By introducing the geometric axiomatic of real number multiplication, we hope to rescue students and teachers from a

similar disconnect, most recently manifested in arguments whether multiplication is repeated addition (see [17]). The lack of an easy visualization of multiplication (how to multiply two line segments), in contrast with addition, may account for some of the struggles that students encounter when they first make the transition from the whole numbers to the integers and rational numbers. Extending multiplication of whole numbers to, integers, rational and real numbers within the same model we hope will be of some pedagogical value.

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